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THE BIHOLOMORPHIC CURVATURE OF QUASISYMMETRIC SIEGEL DOMAINS

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0. Introduction

Let D be a bounded homogeneous domain in $\mathbb{C}^N$ equipped with the canonical Kähler-Einstein Bergman metric. Then D has nonpositive sectional curvature if and only if D is symmetric [6]. For complex manifolds natural generalizations of the sectional curvature are the holomorphic sectional curvature and the holomorphic bisectional curvature as defined in [10]. In this paper we want to investigate the holomorphic bisectional curvature for quasisymmetric Siegel domains, a class of homogeneous Siegel domains which lies strictly between the classes of symmetric domains and general bounded homogeneous domains. Perhaps the simplest characterization of the irreducible quasisymmetric domains is that their Bergman metric induces a symmetric metric on the tube subdomain [4]. A formula for the holomorphic bisectional curvature for quasisymmetric Siegel domains has been given in [15]—using the classification of quasisymmetric Siegel domains and a case-by-case argument. It was shown in [15], however, only that the holomorphic sectional curvature is nonpositive for quasisymmetric Siegel domains.

In the present paper we give a classification free proof of Zelow's formula for the holomorphic bisectional curvature and—as its main result—show that the holomorphic bisectional curvature of quasisymmetric Siegel domains is always nonpositive (§4.14). We would like to note that this is in contrast with a recent paper of Mok-Zhong [11] which states that a *compact* Kähler-Einstein manifold of nonnegative holomorphic bisectional curvature and positive Ricci curvature is isometric to a Hermitian symmetric space. Thus there is no direct analog of the Mok-Zhong result in the noncompact case.

The plan of this paper is as follows. One observes (in §4) that all irreducible quasisymmetric Siegel domains of rank greater than 2 occur as Kähler submanifolds of symmetric Siegel domains (in fact, as fibers of a

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fibration which arises by representing the symmetric domain as a Siegel domain of type three). We also use a standard argument [10] to show that the holomorphic bisectional curvature of the quasisymmetric fiber is majorized by that of the ambient symmetric space and so is nonpositive. This settles the case rank ≥ 3. For the rank 2 case, we use a formula of Zelow [15] and direct computations in the canonical nonassociative algebra associated with the Siegel domain [8].

Since Zelow's formula was proved by a case-by-case argument, we give (in §3) a new unified proof and in the process find a number of new relations in the quasisymmetric case which should be valuable in the future (§2).

1. Basic notation and results

1.1. Let $D = \{(Z, \mathbf{U}) \in \mathbf{V}^{\mathbb{C}} \times \mathbf{U} : \operatorname{Im} Z - F(\mathbf{U}, \mathbf{U}) \in \Omega\}$ be the homogeneous Siegel domain determined by the homogeneous regular cone Ω in the real (finite-dimensional) vector space $\mathbf{V}$, the complex (finite-dimensional) vector space $\mathbf{U}$, and the Ω-hermitian form $F: \mathbf{U} \times \mathbf{U} \to \mathbf{V}^{\mathbb{C}} = \mathbf{V} \oplus i\mathbf{V}$. Let G be the identity component of the automorphism group of D with Lie algebra $\mathfrak{g}$. In G, there is a simply-transitive solvable subgroup S acting by affine transformations on $(\mathbf{V} \oplus i\mathbf{V}) \times \mathbf{U}$; after choosing a base point $b \in D$, we can identify S with D by the map $S \to D$, $g \to g \cdot b$. This identifies the Lie algebra $\mathfrak{s}$ with $T_b D$, and we may assume $\mathfrak{s}$ is a normal j-algebra with respect to the pullback j of the complex structure and the inner product $\langle\ ,\ \rangle$ induced by the metric. We have $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ (orthogonal vector space direct sum), where $\mathfrak{a}$ is abelian, $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$, and the adjoint representation of $\mathfrak{a}$ on $\mathfrak{n}$ has only real eigenvalues. We have a root space decomposition $\mathfrak{n} = \sum \mathfrak{n}_\alpha$, $\mathfrak{n}_\alpha = \{X \in \mathfrak{n} : [H, X] = \alpha(H)X$ for $H \in \mathfrak{a}\}$, and $j\mathfrak{a} \subset \mathfrak{n}$ is the sum of r one-dimensional root spaces, $j\mathfrak{a} = \sum_{k=1}^{r} \mathfrak{n}_{\varepsilon_k}$, where r is the rank of D. All other roots are then of the form $\frac{1}{2}\varepsilon_k$, $\frac{1}{2}(\varepsilon_k \pm \varepsilon_m)$, $k < m$, and we set $\mathfrak{s}_{-1} = \sum_{k \leq m} \mathfrak{n}_{(\varepsilon_k + \varepsilon_m)/2}$, $\mathfrak{s}_{-1/2} = \sum \mathfrak{n}_{\varepsilon_k/2}$, $\mathfrak{s}_0 = j\mathfrak{s}_{-1} = \mathfrak{a} \oplus \sum_{k<m} \mathfrak{n}_{(\varepsilon_k - \varepsilon_m)/2}$.

1.2. It is easy to see that $\mathfrak{s}_{-1}$ is an abelian ideal of $\mathfrak{s}$, whence $\mathfrak{s}$ operates on $\mathfrak{s}_{-1}$ via the adjoint action. We extend this action $\mathbb{C}$-linearly to the complexification $\mathfrak{s}_{-1} \oplus i\mathfrak{s}_{-1}$ of $\mathfrak{s}_{-1}$.

Following [9] we consider the representation $Y \to \phi_Y$ of $\mathfrak{s}$ by affine transformations on $(\mathfrak{s}_{-1} \oplus i\mathfrak{s}_{-1}) \times \mathfrak{s}_{-1/2}$ which is given by

$$\phi_Y(Z, U) = ([jl_2, Z] + (1/2)[u, U] - (i/2)[u, jU] + l_1, [jl_2, U] + u),$$

where $Y = l_1 + jl_2 + u \in \mathfrak{s}$, $l_1, l_2 \in \mathfrak{s}_{-1}$, $u \in \mathfrak{s}_{-1/2}$ and $(Z, U) \in (\mathfrak{s}_{-1} \oplus i\mathfrak{s}_{-1}) \times \mathfrak{s}_{-1/2}$. This induces an affine action of S on $(\mathfrak{s}_{-1} \oplus i\mathfrak{s}_{-1}) \times \mathfrak{s}_{-1/2}$, and there is a natural identification of $\mathfrak{s}_{-1}$ with $\mathbf{V}$ and of $\mathfrak{s}_{-1/2}$ with $\mathbf{U}$ so that the S actions are equivariant. With this identification, $F(u, v) = \frac{1}{4}\{[jU, V] + i[U, V]\}$, $b = (i\sum X_k, 0)$ where $X_k \in \mathfrak{n}_{\varepsilon_k}$ is determined by $\varepsilon_k(j_m X) = \delta_{km}$, and the cone Ω is the orbit of $\sum X_k$ by the action of $\exp \mathfrak{s}_0$ on $\mathbf{V}$.

1.3. Now $\mathbf{V}$ has an algebra structure defined in terms of the Bergman kernel function [8]. Let ∇ be the covariant derivative of the left-invariant metric on S, which is the pull-back of the Bergman metric by the correspondence $S \simeq D$. Using the identification of $\mathfrak{s}_{-1}$ with $\mathbf{V}$, we can express the product as follows: $X \cdot Y = -j\nabla_X Y$ for $X, Y \in \mathfrak{s}_{-1}$ [3]. Let $\mathbf{L}$ denote $\mathbf{V}$ (or $\mathfrak{s}_{-1}$) with this product. We note that the elements $d_k \in \mathbf{V}$ corresponding to X_k form a complete set of primitive orthogonal idempotents [5], and that the corresponding Peirce decomposition agrees with the root space decomposition.

1.4. The notion of a quasisymmetric domain was originally defined by Satake [13]; Dorfmeister [7] showed that this was equivalent to the condition that $\mathbf{L}$ be a Jordan algebra. D'Atri and Miatello [6] proved that an irreducible domain is quasisymmetric if and only if there are constant a and b so that $\dim n_{(\varepsilon_k+\varepsilon_m)/2} = b$, $1 \le k < m \le r$, and $\dim n_{\varepsilon_k/2} = a$, $1 \le k \le r$.

1.5. We note that $\nabla_X Y$ is given by

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle + \langle [Z, Y], X\rangle. \tag{1.5.1}$$

From this it follows easily with $\mathfrak{s}' = \mathfrak{s}_{-1} + \mathfrak{s}_0$

$$\nabla_X Y \in \mathfrak{s}' \quad \text{for all } X, Y \in \mathfrak{s}'. \tag{1.5.2}$$

More precisely, an evaluation of (1.5.1) shows

Lemma. *Let* $\nu, \mu = 0, -\frac{1}{2}, -1$. *Then*

$$\nabla_{\mathfrak{s}_\nu} \mathfrak{s}_\mu \subset \mathfrak{s}_{\nu+\mu} \tag{1.5.3}$$

if $(\nu, \mu) \ne (-\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, -1)$, $(-1, -\frac{1}{2})$ *and* $(-1, -1)$, *while*

$$\nabla_{\mathfrak{s}_{-1}} \mathfrak{s}_{-1} \subset \mathfrak{s}_0. \tag{1.5.4}$$

In general we have

$$\nabla_{\mathfrak{s}_\nu} \mathfrak{s}_\mu \subset \bigoplus_I \mathfrak{s}_\rho,$$

where $I = \{\alpha \in \{0, -\frac{1}{2}, -1\},\ \alpha \equiv (\nu + \mu) \bmod(1)\}$.

Corollary. $\nabla_{\mathfrak{s}_\nu}\mathfrak{s}_\mu \subset \mathfrak{s}_{-1/2}$, *if* $\nu + \mu \equiv -\frac{1}{2}\,\mathrm{mod}(1)$.

2. Special relations

2.1. Let D be a homogeneous Siegel domain with notation as before. In general, we know that for any $X \in \mathfrak{s}$, ∇_X is a skew-symmetric endomorphism of $(\mathfrak{s}, \langle\ \rangle)$, which commutes with j; but there is no simple general relation between $\nabla_{jX}Y$ and $\nabla_X jY$, unless one has commutativity for some of the vectors involved. Thus for example

$$(2.1.1)\quad \nabla_{jU}Y = \nabla_Y jU = j\nabla_Y U = j\nabla_U Y = \nabla_U jY \qquad \text{for } U \in \mathfrak{s}_{-1/2},\ Y \in \mathfrak{s}_{-1},$$

since $j\mathfrak{s}_{-1/2} = \mathfrak{s}_{-1/2}$ and $[\mathfrak{s}_{-1/2}, \mathfrak{s}_{-1}] = 0$. In this section, we consider more special relations of this type, concluding with some which hold only in the quasisymmetric case.

Recall from §1.2 the relation between the Ω-hermitian form F and the bracket product in $\mathfrak{s}$:

$$(2.1.2)\qquad F(U, U') = \tfrac{1}{4}\{[jU, U'] + i[U, U']\} \quad \text{for } U, U' \in \mathfrak{s}_{-1/2} = \mathbf{U}.$$

An easy consequence (e.g. [2, Formula 8]) of the axioms of a normal j-algebra is

$$(2.1.3)\qquad [jU, U'] = -[U, jU'], \qquad U, U' \in \mathfrak{s}_{-1/2},$$

which is consistent with (2.1.2) and the fact that the complex structure on the vector space $\mathbf{U}$ comes from j on $\mathfrak{s}_{-1/2}$. Also for $U, U' \in \mathfrak{s}_{-1/2}$, $Y \in \mathfrak{s}_{-1}$, we have $2\langle\nabla_U U', Y\rangle = \langle[U, U'], Y\rangle$ since $[\mathfrak{s}_{-1/2}, \mathfrak{s}_{-1}] = 0$, while $2\langle\nabla_U U', jY\rangle = -2\langle\nabla_U jU', Y\rangle = -\langle[U, jU'], Y\rangle = -\langle j[U, jU'], jY\rangle$. These formulas and $\nabla_U U' \in \mathfrak{s}_0 \oplus \mathfrak{s}_{-1}$ follow from (1.5.1). As a consequence we obtain for $U, U' \in \mathfrak{s}_{-1/2}$

$$(2.1.4)\qquad \nabla_U U' = \tfrac{1}{2}\{[U, U'] - j[U, jU']\},$$

$$(2.1.5)\qquad \nabla_{jU} jU' = \nabla_U U'.$$

2.2. The following two results hold for arbitrary homogeneous Siegel domains.

Lemma 1. *Suppose* $Y \in \mathfrak{s}_{-1}$ *and* $(\nabla_Y R)|(\mathfrak{s}_0 \oplus \mathfrak{s}_{-1}) \equiv 0$, *where R is the curvature tensor. Then* $0 = \langle\nabla_Y Y, \nabla_{jY} jY\rangle$.

Proof. We have $X_k \in \mathfrak{n}_{\varepsilon_k}$ defined by the condition $\varepsilon_k(jX_l) = \delta_{kl}$. Let $E = \sum X_k$. Then one has [1], [14]–[17]

$$(2.2.1)\quad \nabla_E|(\mathfrak{s}_0 \oplus \mathfrak{s}_{-1}) = j|(\mathfrak{s}_0 \oplus \mathfrak{s}_{-1}), \quad \nabla_{jE} = 0, \quad \nabla_E|\mathfrak{s}_{-1/2} = \tfrac{1}{2}j|\mathfrak{s}_{-1/2},$$

(2.2.2) $$\operatorname{ad} jE|\mathfrak{s}_{-1} = \mathrm{id}, \qquad \operatorname{ad} jE|\mathfrak{s}_0 = 0,$$

and therefore

$$\begin{aligned} 0 &= (\nabla_Y R)(E, Y)Y \\ &= \nabla_Y(R(E, Y)Y) - R(\nabla_Y E, Y)Y - R(E, \nabla_Y Y)Y - R(E, Y)\nabla_Y Y. \end{aligned}$$

But for any $Y' \in \mathfrak{s}_{-1}$, $R(E, Y') = R(jE, jY') = -\nabla_{[jE, jY']} = 0$ by (2.2.1). Thus

$$\begin{aligned} 0 &= -R(\nabla_Y E, Y)Y - R(E, \nabla_Y Y)Y \\ &= -R(jY, Y)Y - R(jE, \nabla_Y jY)Y \\ &= \nabla_Y \nabla_{jY} Y - \nabla_{jY} \nabla_Y Y - \nabla_{[Y, jY]} Y + \nabla_{[jE, \nabla_Y jY]} Y \\ &= 2\nabla_Y \nabla_{jY} Y - \nabla_{jY} \nabla_Y Y, \end{aligned}$$

where the last equality follows from the fact that $\nabla_Y jY \in \mathfrak{s}_{-1}$ implies $-[Y, jY] + [jE, \nabla_Y jY] = \nabla_{jY} Y$, while $[\nabla_{jY} Y, Y] = 0$ implies $\nabla_{\nabla_{jY} Y} Y = \nabla_Y \nabla_{jY} Y$. Hence

$$\begin{aligned} 0 &= 2\langle \nabla_Y \nabla_{jY} Y, jY\rangle - \langle \nabla_{jY} \nabla_Y Y, jY\rangle \\ &= -2\langle \nabla_{jY} Y, \nabla_Y jY\rangle + \langle \nabla_Y Y, \nabla_{jY} jY\rangle \\ &= 3\langle \nabla_Y Y, \nabla_{jY} jY\rangle. \end{aligned}$$

Lemma 2. *For any $A, B \in \mathfrak{s}_{-1}$, one has*

(2.2.3) $$-2\langle \nabla_{jA} jB, \nabla_A B\rangle - \langle \nabla_{jA} jA, \nabla_B B\rangle + 2\langle \nabla_{jB} jA, \nabla_B A\rangle + \langle \nabla_{jB} jB, \nabla_A A\rangle = 0.$$

Proof. First we note that

$$\langle \nabla_{[jA, B]} B, jA\rangle = -\langle [jA, B], \nabla_B jA\rangle = -\langle \nabla_{jA} B, \nabla_B jA\rangle + \langle \nabla_B jA, \nabla_B jA\rangle,$$

so that

$$\begin{aligned} \langle R(jA, B)B, jA\rangle &= -\langle \nabla_B B, \nabla_{jA} jA\rangle + \langle \nabla_{jA} B, \nabla_B jA\rangle - \langle \nabla_{[jA, B]} B, jA\rangle \\ &= -\langle \nabla_B B, \nabla_{jA} jA\rangle + 2\langle \nabla_{jA} B, \nabla_B jA\rangle - \langle \nabla_B A, \nabla_B A\rangle. \end{aligned}$$

Also

$$\langle \nabla_{[A, jB]} jB, A\rangle = \langle [A, jB], \nabla_B jA\rangle = \langle \nabla_A jB, \nabla_B jA\rangle - \langle \nabla_{jB} A, \nabla_B jA\rangle,$$

whence

$$\begin{aligned} \langle R(A, jB)jB, A\rangle &= -\langle \nabla_{jB} jB, \nabla_A A\rangle + \langle \nabla_A jB, \nabla_{jB} A\rangle - \langle \nabla_{[A, jB]} jB, A\rangle \\ &= -\langle \nabla_{jB} jB, \nabla_A A\rangle + 2\langle \nabla_A jB, \nabla_{jB} A\rangle - \langle \nabla_A B, \nabla_A B\rangle. \end{aligned}$$

Since $R\langle (jA, B)B, jA\rangle = \langle R(A, jB)jB, A\rangle$, comparing the above expressions proves the lemma.

2.3. We specialize the results of the last section to quasisymmetric Siegel domains and obtain

Proposition. *For a quasisymmetric domain and any* $A, B \in \mathfrak{s}_{-1}$, *one has*

$$0 = \langle \nabla_A A, \nabla_{jB} jB \rangle + 2\langle \nabla_A B, \nabla_{jB} jA \rangle. \tag{2.3.1}$$

Proof. For an irreducible quasisymmetric domain, the tube subdomain is a totally geodesic Riemannian symmetric submanifold with normal j-algebra $\mathfrak{s}_{-1} \oplus \mathfrak{s}_0$. Hence for all $Y \in \mathfrak{s}_{-1}$, $(\nabla_Y R)|\mathfrak{s}_{-1} \oplus \mathfrak{s}_0 = 0$. Applying Lemma 1 of §2.2 to $Y = A + B$ and to $Y = A - B$ gives

$$0 = \langle \nabla_{A+B}(A+B), \nabla_{j(A+B)} j(A+B) \rangle, \tag{2.3.2}$$

$$0 = \langle \nabla_{A-B}(A-B), \nabla_{j(A-B)} j(A-B) \rangle. \tag{2.3.3}$$

Expanding (2.3.2) and (2.3.3) and adding give

$$\begin{aligned} 0 = \langle \nabla_A A, \nabla_{jB} jB \rangle &+ 2\langle \nabla_A B, \nabla_{jA} jB \rangle \\ &+ 2\langle \nabla_A B, \nabla_{jB} jA \rangle + \langle \nabla_B B, \nabla_{jA} jA \rangle. \end{aligned} \tag{2.3.4}$$

Adding (2.2.3) to (2.3.4) yields the result.

2.4. The following result is important for the description of the holomorphic bisectional curvature.

Lemma. *For an irreducible quasisymmetric domain and* $U, V, W \in \mathfrak{s}_{-1/2}$, $Y, A, B \in \mathfrak{s}_{-1}$, *one has*

$$\langle \nabla_{[U,jU]} jY, Y \rangle = 2\langle \nabla_U U, \nabla_Y Y \rangle = 4\langle \nabla_U Y, \nabla_U Y \rangle, \tag{2.4.1}$$

$$\langle \nabla_V A, \nabla_W B \rangle + \langle \nabla_W A, \nabla_V B \rangle = \langle \nabla_V W, \nabla_A B \rangle, \tag{2.4.2}$$

$$2\langle \nabla_U jY, \nabla_{jY} U \rangle = \langle \nabla_U U, \nabla_{jY} jY \rangle. \tag{2.4.3}$$

Proof. Since the domain under consideration is quasisymmetric, one has from [7] that the fundamental representation ϕ satisfies

$$\phi(Y^2) = (\phi(Y))^2. \tag{2.4.4}$$

From [3], one has in general

$$\phi(Y) = -2j \circ \nabla_Y|\mathfrak{s}_{-1/2}, Y^2 = -j\nabla_Y Y. \tag{2.4.5}$$

Thus (2.4.4) becomes

$$\nabla_Y \nabla_U Y = \nabla_Y \nabla_Y U = \tfrac{1}{2} \nabla_U \nabla_Y Y. \tag{2.4.6}$$

Using (2.1.4) and (2.4.6) we obtain

$$\begin{aligned} \langle \nabla_{[U,jU]} jY, Y \rangle &= -\langle \nabla_Y [U, jU], jY \rangle = \langle [U, jU], j\nabla_Y Y \rangle = 2\langle j\nabla_U U, j\nabla_Y Y \rangle \\ &= -2\langle U, \nabla_U \nabla_Y Y \rangle = -4\langle U, \nabla_Y \nabla_U Y \rangle = 4\langle \nabla_U Y, \nabla_U Y \rangle, \end{aligned}$$

proving (2.4.1). The formula (2.4.2) is obtained from (2.4.1) by polarizing twice.

Now, following the same kind of calculation as in the proof of Lemma 2 in §2.2, one computes

$$\langle \nabla_{[U,jY]} jY, U\rangle = \langle \nabla_U jY, \nabla_Y jU\rangle - \langle \nabla_{jY} U, \nabla_Y jU\rangle,$$

whence

$$\begin{aligned} &\langle R(U, jY)jY, U\rangle \\ (2.4.7) \qquad &= -\langle \nabla_{jY} jY, \nabla_U U\rangle + \langle \nabla_U jY, \nabla_{jY} U\rangle - \langle \nabla_{[U,jY]} jY, U\rangle \\ &= -\langle \nabla_{jY} jY, \nabla_U U\rangle + 2\langle \nabla_U jY, \nabla_{jY} U\rangle - \langle \nabla_U Y, \nabla_U Y\rangle. \end{aligned}$$

Moreover, $\langle R(jU, Y)Y, jU\rangle = -\langle \nabla_Y Y, \nabla_{jU} jU\rangle + \langle \nabla_{jU} Y, \nabla_Y jU\rangle$; thus using (2.4.1) we obtain

$$\langle R(jU, Y)Y, jU\rangle = -\langle \nabla_{jU} Y, \nabla_{jU} Y\rangle. \tag{2.4.8}$$

By (2.1.1) we can drop j here. Finally we use $\langle R(U, jY)jY, U\rangle = \langle R(jU, Y)Y, jU\rangle$ to obtain (2.4.3).

3. The Zelow-Lundquist formula

3.1. In the thesis of Zelow-Lundquist [15, p. 54] and in [16], a formula is given for the holomorphic bisectional curvature of an irreducible quasisymmetric domain. The proof given there is a case-by-case argument (for each of the main classes of quasisymmetric domains). Below we present a classification free proof. Before doing this we explain Zelow's notation and relate it to ours.

Zelow uses vectors in the complex tangent space (and, in addition, has developed what he calls a "more complex notation" [15, p. 40]) while we work in the real tangent space. We identify a vector V in the real tangent space $T_b D$ with the complex vector $\hat{V} = \frac{1}{2}(V - iV)$ of type $(1, 0)$ (see [10, volume II, p. 129] and [15, p. 5]). For any $X \in \mathfrak{s}$, $s \in S$, and point $p = sb \in D$, we have two associated vectors in $T_p D$; one coming from the action of S on D and given by (3.1.1) below and one coming from the identification of S with D and given by (3.1.2) below:

$$X^*|_p = \frac{d}{dt}\bigg|_{t=0} (\exp tX \cdot sb); \tag{3.1.1}$$

$$X^{\#}|_p = \frac{d}{dt}\bigg|_{t=0} (s \exp tX)b. \tag{3.1.2}$$

Of course, at the base point b, these agree and give an unambiguous identification of $\mathfrak{s}$ with T_bD. Next, since D is an open domain of $\mathbf{V}^{\mathbb{C}} \oplus \mathbf{U}$, which can be treated as a real vector space, we have an identification of T_bD with $\mathbf{V}^{\mathbb{C}} \oplus \mathbf{U}$. Combining the identifications $\mathfrak{s} \xrightarrow{\sim} T_bD \xrightarrow{\sim} \mathbf{V}^{\mathbb{C}} \oplus \mathbf{U}$, we get the identifications of $\mathfrak{s}_{-1}$ with $\mathbf{V}$ (or $\mathbf{L}$) and $\mathfrak{s}_{-1/2}$ with $\mathbf{U}$ mentioned in the first section (see [5, pp. 298–9], [3, pp. 14–15]). Choose an orthonormal basis of $\mathfrak{s}$ (it will be useful to assume this basis consistent with the decomposition $\mathfrak{a} \oplus \sum \mathfrak{n}_\alpha$), and use this to define a real coordinate system on $\mathbf{V}^{\mathbb{C}} \oplus \mathbf{U}$, in particular, correspondences between $\mathbb{R}^n$ and $\mathbf{V}$ and between $\mathbb{C}^m$ and $\mathbf{U}$. To avoid confusion, for $a \in \mathbb{R}^n$ and $d \in \mathbb{C}^m$, we let $X_{-1}[a] \in \mathfrak{s}_{-1}$ and $X_{-1/2}[d] \in \mathfrak{s}_{-1/2}$ denote the corresponding elements (compare [5, p. 298]). Then, for $a_1, a_2 \in \mathbb{R}^n$, $d \in \mathbb{C}^m$, the complex vector denoted as $(a_1 + ia_2)\partial_z + d\partial_u$ by Zelow and the element $(X_{-1}[a_1], jX_{-1}[a_2], X_{-1/2}[d]) \in \mathfrak{s}_{-1} \times \mathfrak{s}_0 \times \mathfrak{s}_{-1/2}$ correspond to the same real vector at b. This gives a consistent identification of $\mathbb{C}^n \times \mathbb{C}^m$ with $\mathfrak{s}$.

Zelow, following Satake, has a product structure on $\mathbb{R}^n$. With respect to the identifications of $\mathbb{R}^n$, $\mathbf{V}$, and $\mathfrak{s}_{-1}$, this is the same used in [5] and [3]. In [3], this product is written in terms of the covariant derivative. Zelow also extends the product $\mathbb{C}$-linearly to $\mathbb{C}^n$. Taking all this into account, we have for $a = a_1 + ia_2$, $a' = a_1' + ia_2'$, $a_1, a_2, a_1', a_2' \in \mathbb{R}^n$, that the real vector at b corresponding to $(a \circ a')\partial_z$ is identified with

$$(3.1.3) \qquad -j\nabla_{X_{-1}[a_1]}X_{-1}[a_1'] + j\nabla_{X_{-1}[a_2]}X_{-1}[a_2'] + \nabla_{X_{-1}[a_1]}X_{-1}[a_2'] + \nabla_{X_{-1}[a_2]}X_{-1}[a_1'].$$

Again following Satake, Zelow defines a linear map $a \mapsto R_a$ of $\mathbb{R}^n$ into the space of endomorphisms of $\mathbb{C}^m$ and extends $\mathbb{C}$-linearly to $\mathbb{C}^n$. With respect to the previous identifications, this is 1/2 the map described in [5] (and called there ϕ) and [3]. One finds that for $a = a_1 + ia_2 \in \mathbb{C}^n$, $d \in \mathbb{C}^m$, $R_a d$ corresponds to

$$(3.1.4) \qquad -\nabla_{X_{-1}[a_1]}jX_{-1/2}[d] + \nabla_{X_{-1}[a_2]}X_{-1/2}[d],$$

which, by (2.4.1), becomes

$$(3.1.5) \qquad \nabla_{X_{-1/2}[d]}(X_{-1}[a_2] - jX_{-1}[a_1]).$$

Finally, Zelow uses an inner product, which we will denote by $\{\ ,\ \}$, on $\mathbb{C}^n$. This is defined by extending $\mathbb{C}$-linearly [15, p. 10] the form on $\mathbb{R}^n$ obtained from the Bergman metric at the base point b after identifying $\mathbb{R}^n$ with a subspace of T_bD [15, pp. 26–28]. Thus, after identifying $\mathbb{R}^n$ with $\mathfrak{s}_{-1}$, $\{\ ,\ \}$ is the $\mathbb{C}$-linear extension of our inner product $\langle\ ,\ \rangle$ (determining the left invariant metric on S). Hence, for given $a = a_1 + ia_2$, $a' = a'_1 + ia'_2 \in \mathbb{C}^n$,

$$\begin{aligned}\{a, a'\} &= \langle X_{-1}[a_1], X_{-1}[a'_1]\rangle - \langle X_{-1}[a_2], X_{-1}[a'_2]\rangle \\ &\quad + i(\langle X_{-1}[a_2], X_{-1}[a'_1]\rangle + \langle X_{-1}[a_1], X_{-1}[a'_2]\rangle),\end{aligned} \tag{3.1.6}$$

$$\begin{aligned}&\langle X_{-1}[a_1] + jX_{-1}[a_2], X_{-1}[a'_1] + jX_{-1}[a'_2]\rangle \\ &\quad = \langle X_{-1}[a_1], X_{-1}[a'_1]\rangle + \langle X_{-1}[a_2], X_{-1}[a'_2]\rangle.\end{aligned} \tag{3.1.7}$$

3.2. This section is devoted to translating Zelow's formula for the holomorphic bisectional curvature at b into our language. We note that in [15], the element $e \in \mathbb{R}^n$ corresponds to $E = \sum X_k \in \mathbf{V} = \mathfrak{s}_{-1}$, and the base point in both [15] and [5] is $b = (iE, 0)$. In the following, we use the same symbol F for Zelow's function $\mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n$ and our corresponding function $\mathbf{U} \times \mathbf{U} \to \mathbf{V}^{\mathbb{C}}$ given by (2.1.2). We will also identify vectors $a\partial_z + d\partial_u$ with $X = Y + U + Z$ where $Y = X_{-1}[a_1]$, $U = X_{-1/2}[d]$ and $Z = jX_{-1}[a_2]$.

With these conventions we show

Lemma. *For unit vectors we have*

$$\begin{aligned}&\tfrac{1}{4}(\{a \circ \bar{a}', \bar{a} \circ a'\} + \{a \circ \bar{a}, a' \circ \bar{a}'\} - \{a \circ a', \bar{a} \circ \bar{a}'\}) \\ &\qquad + 4\,\mathrm{Re}\{e, F(R_{\bar{a}}d, R_{\bar{a}'}d')\} + 2\{e, F(R_{\bar{a}}d', R_{\bar{a}}d')\} \\ &\qquad + 2\{e, F(R_{\bar{a}'}d, R_{\bar{a}'}d)\} + 2\{F(d,d), F(d',d')\} \\ &\qquad + 2\{F(d,d'), F(d',d)\} \\ &\quad = \tfrac{1}{4}(\|\nabla_Y Y' + \nabla_{jZ} jZ'\|^2 + \|\nabla_Y jZ' - \nabla_{jZ} Y'\|^2 \\ &\qquad + \langle \nabla_Y Y + \nabla_{jZ} jZ, \nabla_{Y'} Y' + \nabla_{jZ'} jZ'\rangle - \|\nabla_Y Y' - \nabla_{jZ} jZ'\|^2 \\ &\qquad - \|\nabla_{jZ} Y' + \nabla_Y jZ'\|^2 \\ &\qquad + 4\langle \nabla_U (Y - Z), \nabla_{U'}(Y' - Z')\rangle + 2\|\nabla_{U'}(Y - Z)\|^2 \\ &\qquad + 2\|\nabla_U (Y' - Z')\|^2 + 2\langle \nabla_U U, \nabla_{U'} U'\rangle \\ &\qquad + \tfrac{1}{2}\|[jU, U']\|^2 + \tfrac{1}{2}\|[U, U']\|^2).\end{aligned}$$

Remark. "Zelow's formula" asserts that the first expression in the lemma above is—up to a negative constant multiple—the holomorphic bisectional curvature for the unit vectors $a\partial_z + d\partial_u$ and $a'\partial_z + d'\partial_u$.

Proof. An easy computation using (2.2.1) gives

$$\begin{aligned}\langle E, [U, U']\rangle &= \langle E, \nabla_U U' - \nabla_{U'} U\rangle = -\langle \nabla_U E, U'\rangle + \langle \nabla_{U'} E, U\rangle \\ &= -\tfrac{1}{2}\langle jU, U'\rangle + \tfrac{1}{2}\langle jU', U\rangle \\ &= \langle U, jU'\rangle \qquad \text{for } U, U' \in \mathfrak{s}_{-1/2} = \mathbf{U}.\end{aligned} \tag{3.2.1}$$

Thus for $d, d' \in \mathbb{C}^m$

$$\begin{aligned}(3.2.2)\qquad & 4\{e, F(d,d')\} \\ &= \langle E, [jX_{-1/2}[d], X_{-1/2}[d']]\rangle + i\langle E, [X_{-1/2}[d], X_{-1/2}[d']]\rangle \\ &= \langle X_{-1/2}[d], X_{-1/2}[d']\rangle + i\langle X_{-1/2}[d], jX_{-1/2}[d']\rangle.\end{aligned}$$

Now for $a = a_1 + ia_2$, $a' = a'_1 + ia'_2 \in \mathbb{C}^n$, $d, d' \in \mathbb{C}^m$, (3.2.2) implies with (3.1.5),

$$\begin{aligned}(3.2.3)\quad 4\{e, F(R_{\bar a}d, R_{\bar a'}d')\} &= \langle \nabla_{X_{-1/2}[d]}(X_{-1}[a_2] + jX_{-1}[a_1]), \\ &\qquad \nabla_{X_{-1/2}[d']}(X_{-1}[a'_2] + jX_{-1}[a'_1])\rangle \\ &\quad + i\langle \nabla_{X_{-1/2}[d]}(X_{-1}[a_2] + jX_{-1}[a_1]), \\ &\qquad \nabla_{X_{-1/2}[d']}(X_{-1}[a'_2] + jX_{-1}[a'_1])\rangle.\end{aligned}$$

In particular, if $a = a', d = d'$, then (3.2.3) is real.

Next, for $d, d' \in \mathbb{C}^m$, (2.1.2) and (2.1.4) imply

$$\begin{aligned}(3.2.4)\qquad & \{F(d,d), F(d',d')\} \\ &= \tfrac{1}{16}\langle [jX_{-1/2}[d], X_{-1/2}[d]], [jX_{-1/2}[d'], X_{-1/2}[d']]\rangle \\ &= \tfrac{1}{4}\langle \nabla_{X_{-1/2}[d]}X_{-1/2}[d], \nabla_{X_{-1/2}[d']}X_{-1/2}[d']\rangle.\end{aligned}$$

Also, $F(d,d') = a = a_1 + ia_2 = \overline{F(d',d)} \in \mathbb{C}^n$ with

$$X_{-1}[a_1] = \tfrac{1}{4}[jX_{-1/2}[d], X_{-1/2}[d']],\ \ X_{-1}[a_2] = \tfrac{1}{4}[X_{-1/2}[d], X_{-1/2}[d']].$$

Thus by (3.1.6),

$$\begin{aligned}(3.2.5)\qquad 16\{F(d,d'), F(d',d)\} &= \|[jX_{-1/2}[d], X_{-1/2}[d']]\|^2 \\ &\quad + \|[X_{-1/2}[d], X_{-1/2}[d']]\|^2.\end{aligned}$$

Finally, for $a, a' \in \mathbb{C}^n$ arbitrary, we have

$$a \circ a' = (a_1 \circ a'_1 - a_2 \circ a'_2) + i(a_2 \circ a'_1 + a_1 \circ a'_2);$$

therefore

$$\begin{aligned}(3.2.6)\qquad & \{a \circ a', \bar a \circ \bar a'\} \\ &= \|X_{-1}[a_1 \circ a'_1 - a_2 \circ a'_2]\|^2 + \|X_{-1}[a_2 \circ a'_1 + a_1 \circ a'_2]\|^2 \\ &= \|\nabla_{X_{-1}[a_1]}X_{-1}[a'_1] - \nabla_{X_{-1}[a_2]}X_{-1}[a'_2]\|^2 \\ &\quad + \|\nabla_{X_{-1}[a_2]}X_{-1}[a'_1] + \nabla_{X_{-1}[a_1]}X_{-1}[a'_2]\|^2,\end{aligned}$$

$$\begin{aligned}(3.2.7)\qquad & \{a \circ \bar a, a' \circ \bar a'\} \\ &= \langle X_{-1}[a_1 \circ a_1 + a_2 \circ a_2], X_{-1}[a'_1 \circ a'_1 + a'_2 \circ a'_2]\rangle \\ &= \langle \nabla_{X_{-1}[a_1]}X_{-1}[a_1] + \nabla_{X_{-1}[a_2]}X_{-1}[a_2], \nabla_{X_{-1}[a'_1]}X_{-1}[a'_1] \\ &\quad + \nabla_{X_{-1}[a'_2]}X_{-1}[a'_2]\rangle.\end{aligned}$$

Using (3.2.3) to (3.2.7) the assertion follows easily.

Remark. Note that in proving the lemma we have not used any special properties of quasisymmetric domains.

3.3. In the following sections we describe the holomorphic bisectional curvature and show that it is given—up to a negative multiple—by the formula in Lemma 3.2.

First we note that the holomorphic bisectional curvature determined by the unit vectors X and X' is given by

$$(3.3.1)\quad \langle R(X,jX)jX',X'\rangle = -\langle \nabla_{jX}jX', \nabla_X X'\rangle + \langle \nabla_X jX', \nabla_{jX}X'\rangle - \langle \nabla_{[X,jX]}jX', X'\rangle$$

and is determined independent of the choice of unit vectors X and X' in the planes $\mathbb{R}X \oplus \mathbb{R}jX$ and $\mathbb{R}X' \oplus \mathbb{R}jX'$, respectively [10, p. 372].

Expanding X and X' relative to $\mathfrak{s} = \mathfrak{s}_{-1} + \mathfrak{s}_{-1/2} + \mathfrak{s}_0$ we obtain

Lemma. (a) $R(\mathfrak{s}_\nu, \mathfrak{s}_\mu)\mathfrak{s}_\rho \subset \mathfrak{s}_{-1/2}$, *if* $\nu + \mu + \rho \equiv -\frac{1}{2} \bmod(1)$,

(b) $R(\mathfrak{s}_0, \mathfrak{s}_0)\mathfrak{s}_0 \subset \mathfrak{s}_0$, $R(\mathfrak{s}_0, \mathfrak{s}_0)\mathfrak{s}_{-1} \subset \mathfrak{s}_{-1}$,

(c) $R(\mathfrak{s}_0, \mathfrak{s}_{-1/2})\mathfrak{s}_{-1/2} \subsetneq \mathfrak{s}_0 + \mathfrak{s}_{-1}$,

(d) $R(\mathfrak{s}_0, \mathfrak{s}_{-1})\mathfrak{s}_0 \subset \mathfrak{s}_{-1}$, $R(\mathfrak{s}_0, \mathfrak{s}_{-1})\mathfrak{s}_{-1} \subset \mathfrak{s}_0$,

(e) $R(\mathfrak{s}_{-1/2}, \mathfrak{s}_{-1/2})(\mathfrak{s}_0 + \mathfrak{s}_{-1}) \subset \mathfrak{s}_0 + \mathfrak{s}_{-1}$,

(f) $R(\mathfrak{s}_{-1/2}, \mathfrak{s}_{-1})\mathfrak{s}_{-1/2} \subset \mathfrak{s}_0 + \mathfrak{s}_{-1}$,

(g) $R(\mathfrak{s}_{-1}, \mathfrak{s}_{-1})\mathfrak{s}_0 \subset \mathfrak{s}_0$, $R(\mathfrak{s}_{-1}, \mathfrak{s}_{-1})\mathfrak{s}_{-1} \subset \mathfrak{s}_{-1}$.

Proof. The assertion follows easily from the results of §1.5.

3.4. Using Lemma 3.3 many terms in an expansion of (3.3.1) vanish. If one also uses the special properties of the curvature tensor and j operator in a Kähler space, one obtains, by noting that $X = Y + U + Z$ and $X' = Y' + U' + Z'$ as in 3.2,

Lemma. $\langle R(X,jX)jX',X'\rangle = R_1 + R_2 + R_3$, *where*

$$(3.4.1)\quad \begin{aligned} R_1 &= 4\langle R(Z,jY)jY',Z'\rangle + \langle R(Z,jZ)jY',Y'\rangle + \langle R(Z',jZ')jY,Y\rangle \\ &\quad + \langle R(Y,jY)jY',Y\rangle + \langle R(Z,jZ)jZ',Z'\rangle, \end{aligned}$$

$$(3.4.2)\quad \begin{aligned} R_2 &= 2\langle R(Z,jY)jU',U'\rangle + 2\langle R(Z',jY')jU,U\rangle + \langle R(Z,jZ)jU',U'\rangle \\ &\quad + \langle R(Z',jZ')jU,U\rangle + \langle R(Y,jY)jU',U'\rangle \\ &\quad + \langle R(Y',jY')jU,U\rangle \\ &\quad + 4\langle R(Y,jU)jU',Y'\rangle + 4\langle R(Y,jU)jU',Z'\rangle \\ &\quad + 4\langle R(Y',jU')jU,Z\rangle + 4\langle R(Z,jU)jU',Z'\rangle, \end{aligned}$$

$$(3.4.3)\quad R_3 = \langle R(U,jU)jU',U'\rangle.$$

Remark. We would like to point out that this formula is still valid for arbitrary homogeneous Siegel domains.

3.5. In this section we will derive more concrete expressions for R_1, R_2 and R_3 under the additional assumption that D is irreducible and quasisymmetric.

Lemma 1.

$$\begin{aligned}-R_1 = &\|\nabla_Y Y' + \nabla_{jZ} jZ'\|^2 + \|\nabla_Y jZ' - \nabla_{jZ} Y'\|^2 \\ &+ \langle \nabla_Y Y + \nabla_{jZ} jZ, \nabla_{Y'} Y' + \nabla_{jZ'} jZ' \rangle \\ &- \|\nabla_Y Y' - \nabla_{jZ} jZ'\|^2 - \|\nabla_{jZ} Y' + \nabla_Y jZ'\|^2.\end{aligned}$$

Proof. Note

$$\begin{aligned}4\langle R(Z, jY)jY', Z'\rangle &= -4\langle R(jZ, Y)jY', Z'\rangle \\ &= -4\langle \nabla_{jZ} jY', \nabla_Y Z'\rangle + 4\langle \nabla_Y jY', \nabla_{jZ} Z'\rangle\end{aligned}$$

since $[jZ, Y] = 0$, while

$$\begin{aligned}\langle \nabla_{[Z,jZ]} jY', Y'\rangle &= -\langle \nabla_{Y'}[Z, jZ], jY'\rangle \\ &= \langle \nabla_Z jZ, \nabla_{Y'} jY'\rangle - \langle \nabla_{jZ} Z, \nabla_{Y'} jY'\rangle\end{aligned}$$

implies

$$\begin{aligned}&\langle R(Z, jZ)jY', Y'\rangle \\ &\quad = -\langle \nabla_{jZ} jY', \nabla_Z Y'\rangle + \langle \nabla_z jY', \nabla_{jZ} Y'\rangle - \langle \nabla_{[Z,jZ]} jY', Y'\rangle \\ &\quad = 2\langle \nabla_{jZ} Y', \nabla_Z jY'\rangle - \langle \nabla_Z Z, \nabla_{Y'} Y'\rangle + \langle \nabla_{jZ} Z, \nabla_{Y'} jY'\rangle \\ &\quad = \langle \nabla_{jZ} Z, \nabla_{Y'} jY'\rangle\end{aligned}$$

by (2.3.1). From these two formulas we obtain immediately also $\langle R(Z', jZ')jY, Y\rangle = \langle \nabla_{jZ'} Z', \nabla_Y jY\rangle$, $\langle R(Y, jY)jY', Y'\rangle = -\langle \nabla_Y Y, \nabla_{Y'} Y'\rangle$ and $\langle R(Z, jZ)jZ', Z'\rangle = -\langle \nabla_{jZ} Z, \nabla_{jZ'} Z'\rangle$. Adding up the above curvature expressions and completing the appropriate square one obtains the assertion.

Next we consider R_2. We show

Lemma 2.

$$-R_2 = 4\langle \nabla_U(Y - Z), \nabla_{U'}(Y' - Z')\rangle + 2\|\nabla_{U'}(Y - Z)\|^2 + 2\|\nabla_U(Y' - Z')\|^2.$$

Proof. We consider the terms in R_2 separately. First we note that by (2.1.1),

$$\begin{aligned}2\langle R(Z, jY)jU', U'\rangle &= -2\langle R(jZ, Y)jU', U'\rangle \\ &= 2\langle \nabla_Y jU', \nabla_{jZ} U'\rangle - 2\langle \nabla_{jZ} jU', \nabla_Y U'\rangle \\ &= 4\langle \nabla_{U'} Y, \nabla_{U'} Z\rangle,\end{aligned}$$

which also yields

$$2\langle R(Z', jY')jU, U\rangle = 4\langle \nabla_U Y', \nabla_U Z'\rangle.$$

Similarly, by (2.1.1),

$$\begin{aligned}&\langle R(Z, jZ)jU', U'\rangle\\&\quad= -\langle \nabla_{jZ} jU', \nabla_Z U'\rangle + \langle \nabla_Z jU', \nabla_{jZ} U'\rangle - \langle \nabla_{[Z,jZ]} jU', U'\rangle\\&\quad= 2\langle \nabla_{U'} Z, \nabla_Z U'\rangle + \langle \nabla_{jU'} U', \nabla_Z jZ\rangle - \langle \nabla_{jU'} U', \nabla_{jZ} Z\rangle,\end{aligned}$$

which equals, in consequence of (2.4.2) and (2.4.3),

$$2\langle \nabla_{U'} Z, \nabla_Z U'\rangle - 2\langle \nabla_{jU'} Z, \nabla_Z jU'\rangle - 2\langle \nabla_{jU'} jZ, \nabla_{jU'} jZ\rangle.$$

Another application of (2.1.1) finally gives

$$\langle R(Z, jZ)jU', U'\rangle = -2\langle \nabla_{U'} Z, \nabla_{U'} Z\rangle.$$

From these expressions we obtain also

$$\begin{aligned}\langle R(Z', jZ')jU, U\rangle &= -2\langle \nabla_U Z', \nabla_U Z'\rangle,\\ \langle R(Y, jY)jU', U'\rangle &= -2\langle \nabla_{U'} Y, \nabla_{U'} Y\rangle\\ \langle R(Y', jY')jU, U\rangle &= -2\langle \nabla_U Y', \nabla_U Y'\rangle.\end{aligned}$$

Next we have

$$\begin{aligned}4\langle R(Y, jU)jU', Y'\rangle &= -4\langle \nabla_{jU} jU', \nabla_Y Y'\rangle + 4\langle \nabla_Y jU', \nabla_{jU} Y'\rangle\\ &= -4\langle \nabla_U Y, \nabla_{U'} Y'\rangle - 4\langle \nabla_{U'} Y, \nabla_U Y'\rangle + 4\langle \nabla_{U'} Y, \nabla_U Y'\rangle\\ &= -4\langle \nabla_U Y, \nabla_{U'} Y'\rangle,\end{aligned}$$

where in the next to last step we used (2.1.1) and (2.4.2). Similarly, by (2.1.1) and (2.1.4),

$$\begin{aligned}4\langle R(Y, jU)jU', Z'\rangle &= -4\langle \nabla_{jU} jU', \nabla_Y Z'\rangle + 4\langle \nabla_Y jU', \nabla_{jU} Z'\rangle\\ &= 4\langle \nabla_{jU} U', \nabla_Y jZ'\rangle + 4\langle \nabla_{U'} Y, \nabla_U Z'\rangle,\end{aligned}$$

which equals, in consequence of (2.1.1) and (2.4.2),

$$4\langle \nabla_U Y, \nabla_{U'} Z'\rangle - 4\langle \nabla_{U'} Y, \nabla_U Z'\rangle + 4\langle \nabla_{U'} Y, \nabla_U Z'\rangle.$$

Whence

$$4\langle R(Y, jU)jU', Z'\rangle = 4\langle \nabla_U Y, \nabla_{U'} Z'\rangle,$$

which also implies $4\langle R(Y', jU')jU, Z\rangle = 4\langle \nabla_{U'} Y', \nabla_U Z\rangle$. Finally,

$$4\langle R(Z, jU)jU', Z'\rangle = -4\langle R(jZ, U)jU', Z'\rangle = -4\langle \nabla_U Z, \nabla_{U'}, Z'\rangle.$$

Using the above expressions for the summands of R_2 we obtain the assertion.

Finally, we consider R_3. We would like to point out that the following lemma holds for an arbitrary homogeneous Siegel domain.

Lemma 3.

$$-R_3 = -\langle R(U, jU)jU', U'\rangle$$
$$= 2\langle \nabla_U U, \nabla_{U'} U'\rangle + \frac{1}{2}\|[jU, U']\|^2 + \frac{1}{2}\|[U, U']\|^2.$$

Proof. Consider any $U, U' \in \mathfrak{s}_{-1/2}$. Then

$$\begin{aligned}\langle \nabla_{[U,jU]} jU', U'\rangle &= \langle j\nabla_{U'}[U, jU], U'\rangle \\ &= -\langle j\nabla_U jU, \nabla_{U'} U'\rangle + \langle j\nabla_{jU} U, \nabla_{U'} U'\rangle \\ &= 2\langle \nabla_U U, \nabla_{U'} U'\rangle,\end{aligned} \tag{3.5.1}$$

since $[U', [U, jU]] \in [\mathfrak{s}_{-1/2}, \mathfrak{s}_{-1}] = 0$. Also, using (2.1.4) we have

$$\begin{aligned}&\langle R(U, jU)jU', U'\rangle \\ &= -\langle \nabla_{jU} jU', \nabla_U U'\rangle + \langle \nabla_U jU', \nabla_j U'\rangle - \langle \nabla_{[U,jU]} jU', U'\rangle \\ &= -2\langle \nabla_U U', \nabla_U U'\rangle - 2\langle \nabla_U U, \nabla_{U'} U'\rangle,\end{aligned} \tag{3.5.2}$$

which gives

$$\langle \nabla_U U', \nabla_U U'\rangle = \langle \nabla_{U'} U, \nabla_{U'} U\rangle, \tag{3.5.3}$$

since $\langle R(U, jU)jU', U'\rangle = \langle R(U', jU')jU, U\rangle$. Using (3.5.3) and (2.1.4) we obtain

$$\begin{aligned}&\|[jU, U']\|^2 + \|[U, U']\|^2 \\ &= \langle \nabla_{jU} U', \nabla_{jU} U'\rangle - 2\langle \nabla_{jU} U', \nabla_{U'} jU\rangle + \langle \nabla_{U'} jU, \nabla_{U'} jU\rangle \\ &\quad + \langle \nabla_U U', \nabla_U U'\rangle - 2\langle \nabla_U U', \nabla_U U\rangle + \langle \nabla_{U'} U, \nabla_{U'} U\rangle \\ &= 4\langle \nabla_U U', \nabla_U U'\rangle.\end{aligned} \tag{3.5.4}$$

Thus

$$\begin{aligned}&2\langle \nabla_U U, \nabla_{U'} U'\rangle + \tfrac{1}{2}\|[jU, U']\|^2 + \tfrac{1}{2}\|[U, U']\|^2 \\ &\quad = -\langle R(U, jU)jU', U'\rangle = -R_3.\end{aligned} \tag{3.5.5}$$

Using Lemma 3.4 and the three lemmas of this section we have shown

Theorem (*Zelow's formula*). *The holomorphic bisectional curvature of an irreducible quasisymmetric Siegel domain is—up to a negative multiple—given by the formulas of Lemma* 3.2.

4. Holomorphic bisectional curvature

4.1. As mentioned in the introduction, we investigate the holomorphic bisectional curvature of irreducible quasisymmetric Siegel domains in the cases where rank ≥ 3 and rank $= 2$ by different methods.

The following lemma is an essential tool for the rank ≥ 3 case. It is a straightforward generalization of [10, IX, Proposition 9.2].

Lemma. *Let M be a complex submanifold of a Kähler manifold M' with second fundamental form α. Let R and R' be the Riemannian curvature tensor fields of M and M' respectively. Then*

$$\langle R(X,jX)jX',X'\rangle = \langle R'(X,jX)jX',X'\rangle - 2\langle \alpha(X,X'),\alpha(X,X')\rangle$$

for all vector fields X, X' on M.

Corollary. *Let M be a complex submanifold of a Kähler manifold M'. Then M has nonpositive holomorphic bisectional curvature if M' has.*

4.2. In this section we consider the case where rank ≥ 3. Using well-known results (see e.g. [14, II, §§6,7 and I, §5] one obtains

Theorem. *Let $D(\Omega, S)$ be an irreducible quasisymmetric Siegel domain and g its Bergman metric.*

(a) *If* rank $D(\Omega,S) \geq 3$, *then $D(\Omega,S)$ occurs as a fiber in the realization of a symmetric Siegel domain B as a domain of type* III. *The group $\mathscr{A}$ of linear transformations of $D(\Omega,S)$ which are restrictions of automorphisms of B is reductive.*

(b) *The statements of* (a) *also hold for the quasisymmetric Siegel domains for which the corresponding algebra is* $\mathrm{Herm}(2,\mathbb{C})$ *or* $\mathrm{Sym}(2,\mathbb{R})$ *or* $\mathrm{Herm}(2,\mathbb{H})$, *where in the latter case we have to assume that in* $\mathbf{U}$ *only one type of irreducible representation of* $\mathrm{Herm}(2,\mathbb{H})$ *occurs and with proper multiplicity.*

(c) *The Riemannian metric on the Siegel domain $D(\Omega,S)$ considered in* (a) *and* (b) *which is induced from the ambient symmetric space B is a multiple of the Bergman metric.*

(d) *The Siegel domains considered in* (a) *and* (b) *have negative holomorphic bisectional curvature in their Bergman metric.*

Proof. In view of [14] it suffices to prove (c). We note that the metric $\tilde{g}$ induced on $D(\Omega,S)$ from the symmetric space B is a Kähler metric and admits a connected transitive group G of affine holomorphic isometries ([14, III, §5], [7, II, §5]). Moreover, the linear part LG of G contains the connected component of the group of automorphisms of Ω (a reductive group) [14, III, Theorem 2.3]. This shows that G contains a transitive subgroup H such that $\mathrm{Ad}\,h$, $h \in H$, has only real eigenvalues. Then an argument of Gindikin and Vinberg shows that the skew form κ induced on $\mathfrak{h} = \mathrm{Lie}\,H$ from the Kähler form on $D(\Omega,S)$ is actually of the form $\kappa(x,y) = \omega([x,y])$ for some linear form $\omega\colon \mathfrak{h} \to \mathbb{R}$. We write $\mathfrak{g} = \mathrm{Lie}\,G = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0$, where the subscripts denote the negative of the eigenvalues of $\mathrm{ad}\,je$ in $\mathfrak{g}$, e being the maximal idempotent of $\mathfrak{h}$. Moreover, $\mathfrak{k} \subset \mathfrak{g}_0$

denotes the isotropy subalgebra of $ie \in D(\Omega, S)$. Then for $k \in \mathfrak{k}$ and $x \in \mathfrak{g}_{-1}$ we have

$$0 = \kappa(A,x) = \kappa(A,[je,x]) = \kappa(je,[A,x]) = \omega([je,[A,x]]) = \omega([A,x]),$$

where we have used the closedness condition for κ and x, $[A,x] \in \mathfrak{g}_{-1}$ and $A \in \mathfrak{g}_0$. We use the notation of [5, §1] and choose $x = X_{-1}[b_{kl}]^+$ and $A = (L_{d_k}(b_{kl}) - L_{d_1}(b_{kl}), *)$. Then $A \in \mathfrak{k}$, and a direct computation shows $0 = \omega([A,x]) = \alpha\omega(X_k - X_l)$, where α is a nonzero factor.

Hence $\omega(X_k) = \omega(X_l)$ for all k, l. From [6, §6] and [6, Proposition 3] it follows now that ω is a multiple of the "Bergman form ω_g", whence the claim.

4.3. In the remaining sections of this paper we consider irreducible quasisymmetric domains for which the associated (simple formally real Jordan) algebra $\mathbf{L}$ is of rank 2, i.e., there exist a nondegenerate bilinear form μ of signature $(1, \dim \mathbf{L} - 1)$ on $\mathbf{L}$ and an identity element E of $\mathbf{L}$ such that the product in $\mathbf{L}$ is given by $X \cdot Y = \mu(X,E)Y + \mu(Y,E)X - \mu(X,Y)E$.

Opposite to the case where rank ≥ 3, we do not consider a fibration of some symmetric domain, but for a given tangent direction X we consider a quasisymmetric Siegel domain with algebra isomorphic to $\mathrm{Herm}(2,\mathbb{C})$ or $\mathrm{Sym}(2,\mathbb{R})$ which has also X as tangent vector. What makes things work is the fact that the new Siegel domain has nonpositive holomorphic bisectional curvature (by §4.2) and its curvature can be related explicitly to the one of the original domain (see the proof of Theorem 4.14).

In order to be able to do this it is crucial to observe:

(4.3.1) Let $\mathbf{L}' \subset \mathbf{L}$ be a subspace containing E. Then $\mathbf{L}'$ is a subalgebra of $\mathbf{L}$. In particular, if $\dim \mathbf{L}' \geq 3$, then $\mathbf{L}'$ is a simple formally real Jordan algebra of rank 2.

Set $n = \dim \mathbf{L}$ and $n' = \dim \mathbf{L}'$, and denote by K and K' the "domain of positivity" of $\mathbf{L}$ and $\mathbf{L}'$ respectively. Then for the "invariant" of the corresponding cone we have (with some constants $c, c' \in \mathbb{R}$)

$$\iota_K(X) = c\mu(X,X)^{-n/2}, \qquad X \in K, \tag{4.3.2}$$

$$\iota_{K'}(X) = c'\mu(X,X)^{-n'/2}, \qquad X' \in K', \tag{4.3.3}$$

These formulas follow from the fact that both sides have the same transformation property relative to the automorphisms of the cone under consideration (see e.g. formulas (1.7) and (1.8) of [8]).

4.4. We want to "restrict" the Ω-hermitian form F of the given Siegel domain relative to a subalgebra $\mathbf{L}'$ of $\mathbf{L}$. We set

$$\mathbf{F}'\colon \mathbf{U} \times \mathbf{U} \to \mathbf{L}'_{\mathbb{C}}, \qquad F'(U,W) = \mathrm{proj}(\mathbf{L}'_{\mathbb{C}})F(U,W).$$

Lemma. *F' is a Ω'-hermitian form.*

Proof. Since the sesquilinearity of F' follows immediately from the corresponding property of F, it suffices to show $F'(U,U) \in \overline{\Omega'}$ for all $U \in \mathbf{U}$ and that $F'(U,U) = 0$ implies $U = 0$. But $F'(U,U) = 0$ just says that $F(U,U)$ is perpendicular to $\mathbf{L}'$ and in particular to E. Using σ as in [5], i.e., σ is the restriction of $\langle \cdot, \cdot \rangle$ to $\mathbf{L}$, and standard properties of σ and F we thus obtain $0 = \sigma(E, F(U,U)) = \rho(U,U)$, whence $U = 0$. To verify $F'(U,U) \in \overline{\Omega'}$ where "$-$" denotes the closure operation, we note that Ω' is a selfdual cone relative to $\tau' = \sigma|\mathbf{L}' \times \mathbf{L}'$. We also note $\overline{\Omega}' = \{x^2; x \in \mathbf{L}'\} \subset \{x^2; x \in \mathbf{L}\} = \overline{\Omega}$. Hence, for $X \in \Omega'$ we have $0 \leq \sigma(X, F(U,U)) = \sigma(X, F'(U,U))$, whence $F'(U,U) \in \overline{\Omega}'$.

Corollary. $D(\Omega', F') = \{(Z', U) \in \mathbf{L}'_{\mathbf{C}} \times \mathbf{U}; \operatorname{Im} Z' - F'(U,U) \in \Omega'\}$ *is an irreducible quasisymmetric Siegel domain if* $\dim L' \geq 3$.

Proof. From the lemma above we know that $D(\Omega', F')$ is a Siegel domain. It is irreducible, since the cone Ω' is irreducible. To prove that it is quasisymmetric it suffices to show that there exists a reductive linear subgroup of $\operatorname{Aut} D(\Omega', F')$ which acts transitively on Ω'. With the notation of [5] we denote by $\mathfrak{g}'$ the Lie algebra generated by $(L(b), \frac{1}{2}\phi(b))$, $b \in \mathbf{L}'$, and denote by G' the corresponding Lie subgroup of linear automorphisms of $D(\Omega, F)$. Set $\tau' = \sigma|\mathbf{L}' \times \mathbf{L}'$. Then for $X \in \mathbf{L}'$, $U, V \in \mathbf{U}$ and $(W, \hat{W}) \in G'$ we have

$$\begin{aligned} \tau'(X, WF'(U,V)) &= \sigma(X, WF(U,V)) = \sigma(X, F(\hat{W}U, \hat{W}V)) \\ &= \tau'(X, F'(\hat{W}U, \hat{W}V)). \end{aligned}$$

Hence $G' \subset \operatorname{Aut} D(\Omega', F')$ and the assertion follows.

4.5. For our final goal we need to investigate the relations between the two quasisymmetric Siegel domains $D(\Omega, F)$ and $D(\Omega', F')$ more closely. We know that the corresponding Bergman kernels are induced from $\eta_{\Omega F}(X) = a\iota_{\Omega}(X)^r$ and $\eta_{\Omega' F'}(X) = a'\iota_{\Omega'}(X)^{r'}$ respectively. (For a definition of $\eta_{\Omega F}$ see [8, §2]. The transformation property stated there implies that $\eta_{\Omega F}$ is a power of the invariant η_{Ω} since $D(\Omega, F)$ is quasisymmetric and irreducible. For an explicit statement see e.g. [7, II, Corollary 1.4]). Using (4.3.2) and (4.3.3) we thus obtain

$$\eta_{\Omega F}(X) = b\eta_{\Omega' F'}(X)^s \quad \text{for } X \in \Omega' \subset \Omega, \tag{4.5.1}$$

where $b > 0$ and $s \geq 1$.

Here the last statement follows from the fact that $\mu(X,X)^{-n/2} = \mu(X,X)^{-n's/2}$ for $X \in \Omega'$ implies $n = n's$, whence $s \geq 1$.

Using the definition of σ and σ' for $D(\Omega, F)$ and $D(\Omega', F')$ from $\eta_{\Omega F}$ and $\eta_{\Omega' F'}$ as in [5] now shows

$$\sigma|\mathbf{L}' \times \mathbf{L}' = s\sigma'. \tag{4.5.2}$$

The definition of ρ in [5] then implies

$$\rho = s\rho'. \tag{4.5.3}$$

To verify this we note $\rho(U, V) = \sigma(E, F(U, V)) = \sigma(E, F'(U, V)) = s\sigma'(E, F'(U, V)) = s\rho'(U, V)$.

Thus for the map ϕ' given by $\sigma'(X, F'(U, V)) = \rho'(\phi'(X)U, V)$ we have

$$\phi' = \phi|\mathbf{L}'. \tag{4.5.4}$$

4.6. In the following sections we discuss the cases $3 \leq \dim \mathbf{L} \leq 6$ separately. We will need this for the proof of our main result at the end of this paper.

The cases $\dim \mathbf{L} = 3, 4$ follow from Theorem 4.2, so we mostly discuss the cases $\dim \mathbf{L} = 5, 6$. Since we assume that our domain is irreducible of rank 2, our domain is determined once we specify $\dim \mathbf{L}$ and the irreducible representations of $\mathbf{L}$ which occur in $\mathbf{U}$ (with multiplicity).

If $\dim \mathbf{L} = 5$, then $\mathbf{L}$ has only one type of irreducible representation, but if $\dim \mathbf{L} = 6$, there are two inequivalent irreducible representations.

If we are in the case $\dim \mathbf{L} = m$, we denote by $\mathbf{D}_m$ the curvature expression which occurs on the left-hand side of the equation in Lemma 3.2.

If $\dim \mathbf{L} = 6$ and we know or assume that only one type of irreducible representation of $\mathbf{L}$ occurs in $\mathbf{U}$, we indicate this by adding the superscript "+", e.g. D_6^+ denotes the corresponding domain and $\mathbf{D}_6^+$ the associated curvature expression.

From Theorem 4.2 we know

$$\mathbf{D}_6^+ \geq 0. \tag{4.6.1}$$

Assume now $\dim \mathbf{L} = 6$ and, for fixed $a, a' \in \mathbf{L}^{\mathbb{C}}$, set

$$\mathbf{L}' = \operatorname{span}\{E, \operatorname{Re}(a), \operatorname{Im}(a), \operatorname{Re}(a'), \operatorname{Im}(a')\}. \tag{4.6.2}$$

Then $\mathbf{L}'$ is a subalgebra of $\mathbf{L}$ and we can assume $\dim \mathbf{L}' = 5$.

An inspection of Lemma 3.2 shows

$$\mathbf{D}_6 = c\mathbf{D}_5 + A, \tag{4.6.3}$$

where $\mathbf{D}_5$ is the curvature expression for the subdomain determined by $\mathbf{L}'$, c is a positive constant determined as in (4.5.2) and (4.5.3), and the remainder is given by

$$A = 2\{F(d, d)^\perp, F(d', d')^\perp\} + 2\{F(d, d')^\perp, F(d', d)^\perp\}. \tag{4.6.4}$$

Here $F(u,v)^{\perp}$ denotes the component of $F(u,v)$ perpendicular to $\mathbf{L}'$.

If we are considering a domain in which only one type of irreducible representation of $\mathbf{L}$ occurs in $\mathbf{U}$, we denote the remainder term in (4.6.4) by A^+ so

$$\mathbf{D}_6^+ = c\mathbf{D}_5 + A^+. \tag{4.6.5}$$

Note that no matter which type of $\mathbf{L}$ we start with, the $\mathbf{D}_5$ term depends only on the multiplicity of the unique representation of $\mathbf{L}'$. That is because the difference between (4.6.4) and (4.6.5) can only occur from the different action of elements of $\mathbf{L}$ which are perpendicular to $\mathbf{L}'$, and this shows only in A and A^+ respectively.

To make this more precise and to evaluate it, we describe the domain D_6^+ associated with $\dim \mathbf{L} = 6$ when there is only one type of representation of $\mathbf{L}$. Note that if all representations are of one type with a fixed multiplicity, then the corresponding domains are equivalent, no matter which of the two inequivalent representations is used.

4.7. We want to describe the domain D_6^+ which corresponds to the case where $\dim \mathbf{L} = 6$ and only one type of irreducible representations occurs in $\mathbf{U}$.

Let $\mathbf{A}_n$ denote the Jordan algebra of $n \times n$ Hermitian matrices with entries in the real division algebra of quaternions. The Jordan algebra product is related to the matrix product by $a \circ a' = \frac{1}{2}(aa' + a'a)$, and the identity element is the identity matrix. Every $a \in \mathbf{A}_n$ has a block decomposition

$$a = \begin{pmatrix} z & u \\ u^* & w \end{pmatrix}, \tag{4.7.1}$$

where $z \in \mathbf{A}_2$, $w \in \mathbf{A}_{n-2}$, and $u \in \mathbf{A}_{1/2} \equiv \mathrm{Mat}(2, n-2; \mathbb{H})$. We also write

$$a = a_1 + a_{1/2} + a_0, \tag{4.7.2}$$

where

$$a_1 = a_1(z) = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}, \quad a_{1/2} = a_{1/2}(u) = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix},$$
$$a_0 = a_0(w) = \begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}.$$

We use similar notation for the complexifications. Let e_1 (resp. e_0) denote the identity element of $\mathbf{A}_2$ (resp., $\mathbf{A}_{n-2}$), and note that the imbeddings $z \to a_1(z)$ and $w \to a_0(w)$ are consistent with the algebra operations. Then D_6^+ can be realized as

$$D_6^+ = \{(z,u) \in (\mathbf{A}_2 \otimes \mathbb{C}) \times (\mathbf{A}_{1/2} \otimes \mathbb{C}) \colon \mathrm{Im}\, z - \tfrac{1}{2}(u\bar{u}^* + \bar{u}a^*) \in Y_2\}, \tag{4.7.3}$$

where Y_k denotes the domain of positivity of $\mathbf{A}_k$, and conjugation is with respect to the complexification of $\mathbf{A}_{1/2}$. Note that here $F(u,v) = \frac{1}{2}(u\bar{v}^* + \bar{v}u^*)$. A straightforward calculation shows that the map

$$\rho(z,u) = a_1(z) - \tfrac{i}{2}a_1(e_1)(a_{1/2}(u))^2 + a_{1/2}(u) + ia_0(e_0),$$

i.e.,

$$\rho(z,u) = \begin{pmatrix} z - \frac{i}{2}uu^* & u \\ u^* & ie_0 \end{pmatrix}, \tag{4.7.4}$$

is a biholomorphism from D_6^+ onto the submanifold B_6 of the symmetric tube domain T_n where

$$\begin{aligned} B_6 &= \{a \in \mathbf{A}_n \otimes \mathbb{C} : \operatorname{Im} a \in Y_n, a_0 = ia_0(e_0)\}, \\ T_n &= \{a \in \mathbf{A}_n \otimes \mathbb{C} : \operatorname{Im} a \in Y_n\}. \end{aligned} \tag{4.7.5}$$

On T_n, we take the multiple of the Bergman metric defined by $\{a, a'\} = 8\operatorname{Re}\operatorname{Tr}(aa')$, $a, a' \in \mathbf{A}_n \otimes \mathbb{C}$, where $\operatorname{Re}\operatorname{Tr}$ denotes the real part of the quaternionic trace in $\mathbb{H}$. Let R_6^+ denote the curvature expression as in Lemma 3.2 for the tube domain T_n. A computation shows

$$\begin{aligned} R_6^+ &= \tfrac{1}{4}(\{a \circ \bar{a}', \bar{a} \circ a'\} + \{a \circ \bar{a}, a' \circ \bar{a}'\} - \{a \circ a', \bar{a} \circ \bar{a}'\}) \\ &= \operatorname{Re}\operatorname{Tr}(a\bar{a}a'\bar{a}' + a'\bar{a}a\bar{a}'), \end{aligned} \tag{4.7.6}$$

which in turn by Theorem 3.5 is just $-\frac{1}{4}\langle R(X, jX)jX', X'\rangle$, where $X = X_{-1}[\operatorname{Re} a] + jX_{-1}[\operatorname{Im} a]$, etc.

4.8. We want to compute the metric induced on B_6 from T_n. To this end we will show that the map ρ given in (4.7.4) is equivariant with respect to a sufficiently large group.

We note that B_6 inherits naturally biholomorphic transformations from T_n:

$$t_{a_1} z = (z_1 + a_1) + z_{1/2} + z_0, \qquad a_1 \in \mathbf{A}_2, \tag{4.8.1}$$

$$t_{a_{1/2}} z = z_1 + (z_{1/2} + a_{1/2}) + z_0, \qquad a_{1/2} \in \mathbf{A}_{1/2}. \tag{4.8.2}$$

It is straightforward to check that these two biholomorphic maps of B_6 induce on D_6^+ the following transformations:

$$t'_{a_1}(z,u) = (z + a_1, u), \tag{4.8.1$'$}$$

$$t'_{a_2}(z,u) = (z + 2iF(u, a_{1/2}) + iF(a_{1/2}, a_{1/2}), u + a_{1/2}), \tag{4.8.2$'$}$$

where we have identified $\mathbf{A}_{1/2} \otimes \mathbb{C}$ with $\mathbf{U}$ by (4.7.2).

Note that (4.8.2)′ is one of the typical biholomorphic maps of D_6^+; however, only *real* $a_{1/2}$ occur here.

Consider B_6 again; we have also linear transformations of T_n that leave B_6 invariant. Since all linear automorphisms g of T_n are of the type $gz = wzw^*$ for some invertible $n \times n$ matrix with quaternionic entries, it suffices to specify w:

$$w = \begin{pmatrix} I & q \\ 0 & I \end{pmatrix}, \tag{4.8.3}$$

$$w = \begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix}. \tag{4.8.4}$$

We note that here q and h have coefficients in $\mathbb{H}$, not in $\mathbb{H}\otimes\mathbb{C}$. However, in the induced transformations on D_6^+ complex entries will occur. This follows from

$$g(z,u) = (z + 2i\tfrac{1}{2}e_1(u\overline{iq}) + \tfrac{i}{2}e_1(q^2),\ u + iq), \tag{4.8.3$'$}$$

$$g(z,u) = (hzh^t,\ hu). \tag{4.8.4$'$}$$

Here the first transformation corresponds to (4.8.2)$'$, but has now purely imaginary "parameter" iq. The second equation describes linear automorphisms of D_6^+. One should remark that actually the $\mathrm{Mat}(2, n-2, \mathbb{H}) \otimes \mathbb{C}$ component of u is multiplied by h from the left.

From (4.8.4)$'$ it follows that the full group of automorphisms of Y_2 acts linearly on D_6^+. Now the argument to prove Theorem 4.2(c) shows

Proposition. *The metric induced from T_n on B_6 is a multiple of the Bergman metric (after identifying D_6^+ with B_6 via ρ).*

Remarks. 1. We would like to point out that the number of irreducible representations of $\mathbf{L} \cong \mathbf{A}_2$ in $\mathbf{U}$ of D_6^+ is *even.* This follows from the fact that $\mathbf{L}$ has $n-2$ irreducible representations in $\mathrm{Mat}(2, n-2, \mathbb{H})$ and that $\mathbf{U} = \mathrm{Mat}(2, n-2, \mathbb{H}) \otimes \mathbb{C}$. It is important to observe that the irreducible representations occurring in the "real part of $\mathbf{U}$" and the "imaginary part of $\mathbf{U}$" are the same as real representations. However, these real irreducible representations are actually already complex representations. Hence complexifying again splits the representations into complex and conjugate complex representations. In the case of $\mathbf{A}_2$, i.e., quaternionic matrices, these two representations are equivalent as complex representations. In case one carries out the same construction with $\mathbf{A} \cong$ complex hermitian matrices, two inequivalent representations will result.

2. As indicated just above, the construction of D_6^+ can be generalized to other formally real Jordan algebras. This has been carried out (even more generally) in [7, Chapter II].

4.9. In this section we want to give an explicit formula for the holomorphic bisectional curvature of D_6^+. First, however, we describe the second fundamental form of B_6 in T_n.

We note that the differential of ρ (as given in (4.7.4)) at the point ie_1+ie_0 is the identity map. Therefore a, a', d, d' in Lemma 3.2 are the same as those used in the computation of $\mathbf{R}_6^+$ with

$$a \longrightarrow \begin{pmatrix} a & d \\ d^* & 0 \end{pmatrix} \quad \text{and} \quad a' \longrightarrow \begin{pmatrix} a' & d' \\ d'^* & 0 \end{pmatrix}.$$

Since T_2 is naturally imbedded in both D_6^+ and B_6, we see that $\{X, Y\} = 8\,\mathrm{Re}\,\mathrm{trace}\,XY$ in Lemma 3.2. Note that here X, Y are 2×2-matrices.

Proposition. *The square of the second fundamental form* II *of B_6 in T_n is given by*

$$\{\mathrm{II}, \mathrm{II}\} = \{e_0(d \circ d'), e_0(\bar{d} \circ \overline{d'})\}. \tag{4.9.1}$$

Proof. We will give a proof that actually applies to all formally real Jordan algebras $\mathbf{A}$. We have to compare

$$\mathbf{R} = \tfrac{1}{4}(\{r \circ \overline{r'}, \bar{r} \circ r'\} + \{r \circ \bar{r}, r' \circ \overline{r'}\} - \{r \circ r', \bar{r} \circ \overline{r'}\})$$

to the expression $\mathbf{D}$ of Lemma 3.2, where $r = a + d$, $r' = a' + d'$ and $a, a' \in \mathbf{A}(e_1) \otimes \mathbf{C}$, $d, d' \in \mathbf{A}_{1/2}(e_1) \otimes \mathbf{C}$. Moreover, we will use $F(u, v) = \frac{1}{2}e_1(u \circ \bar{v})$ for $u, v \in \mathbf{A}_{1/2}(e_1) \otimes \mathbf{C}$, and $R_a d = a \circ d$ for $a \in \mathbf{A}_1(e_1) \otimes \mathbf{C}$, $d \in \mathbf{A}_{1/2}(e_1) \otimes \mathbf{C}$. An expansion of $\mathbf{R}$ shows that $\mathbf{R}$ differs from $\mathbf{D}$ only in terms involving solely d and d'. So for $d, d' \in \mathbf{A}_{1/2}(e_1) \otimes \mathbb{C}$ we have to compute $\mathbf{R}(d, d') - \mathbf{D}(d, d')$. Splitting $d \circ d' = e_1(d \circ d') + e_0(d \circ d')$ etc., we obtain after a somewhat tedious calculation

$$\begin{aligned}\mathbf{R}(d, d') - D(d, d') = {} & \tfrac{1}{4}[\{\bar{d} \circ e_0(d \circ \overline{d'}), d'\} + \{d \circ e_0(\overline{d'} \circ \bar{d}), d'\} \\ & \qquad + \{\overline{d'} \circ e_0(\bar{d} \circ d), d'\}] \\ & - \tfrac{1}{4}[\{\bar{d} \circ e_1(d \circ \overline{d'}), d'\} + \{d \circ e_1(\overline{d'} \circ \bar{d}), d'\} \\ & \qquad + \{\overline{d'} \circ e_1(\bar{d} \circ d), d'\}] \\ & - \tfrac{1}{2}\{e_0(d \circ d'), e_0(\bar{d} \circ \overline{d'})\}.\end{aligned}$$

From general Jordan identities, the two square brackets are equal. Thus, in view of Lemma 4.2, the proposition is proven.

Corollary.

$$\begin{aligned}\mathbf{D}_6^+(a, a') = {} & \mathrm{Re}\,\mathrm{trace}(a\bar{a}a'\overline{a'} + a'\bar{a}a\overline{a'}) \\ & + 4\,\mathrm{Re}\,\mathrm{trace}\, e_0(a_{1/2}a'_{1/2})e_0(\bar{a}_{1/2}\overline{a'_{1/2}}).\end{aligned}$$

4.10. We want to consider D_5. If m irreducible representations of $\mathbf{L}$ occur in $\mathbf{U}$, and m is odd, then we embed D_5 into the corresponding domain with $m+1$ irreducible representations. Since this embedding is totally geodesic, it suffices to consider $\mathbf{D}_5$ for even $m = 2n$.

Let D_6^+ be constructed from $\mathbf{A}_n$ as in the previous sections. Then Corollary 4.9 gives an explicit formula for $\mathbf{D}_6^+$. Moreover, from (4.6.5) we know $\mathbf{D}_6^+ = c\mathbf{D}_5 + A^+$, where A^+ is given by (4.6.4), i.e., $A^+ = 2\{F(d,d)^\perp, F(d',d')^\perp\} + 2\{F(d,d')^\perp, F(d',d)^\perp\}$. Here $\{\cdots\} = 8\,\mathrm{Re\,trace}\cdots$ and $F(d,d') = \frac{1}{2}e_1(d \circ \overline{d'})$, where $d, d' \in \mathbf{A}_{1/2} \otimes \mathbb{C}$.

Altogether we have thus to consider

$$
\begin{aligned}
\mathbf{S} = {} & \mathrm{Re\,trace}(a\bar{a}a'\overline{a'} + a'\bar{a}a\overline{a'}) \\
& + 4\,\mathrm{Re\,trace}(e_0(a_{1/2} \circ a'_{1/2})e_0(\overline{a_{1/2}} \circ \overline{a'_{1/2}})) \\
& - 4\,\mathrm{Re\,trace}(e_1(a_{1/2} \circ \overline{a_{1/2}})^\perp e_1(a'_{1/2} \circ \overline{a'_{1/2}})^\perp) \\
& - 4\,\mathrm{Re\,trace}(e_1(a_{1/2} \circ \overline{a'_{1/2}})^\perp e_1(a'_{1/2} \circ \overline{a_{1/2}})^\perp).
\end{aligned}
\tag{4.10.1}
$$

To compute expressions of type "$p^\perp$" in (4.10.1) we write $\mathbb{H} = \mathbb{R}f_0 + \mathbb{R}f_1 + \cdots + \mathbb{R}f_3$, where $f_0 = 1$ and $f_0, \cdots, f_3$ are the natural generators of $\mathbb{H}$. In particular we have $f_m^2 = -1$. We can assume that $\mathbf{L}'$, as defined in (4.6.2), consists of all 2×2-hermitian matrices with entries in $\mathbb{H} \ominus \mathbb{R}f_3$.

Moreover, from the definition of $\mathbf{L}'$ we derive that we can assume that $x \in \mathrm{Sym}(2,\mathbb{R}) \otimes \mathbb{C}$ holds. Next we expand the term involving e_0. We obtain

$$
\begin{aligned}
& 4\,\mathrm{Re\,trace}\, e_0(a_{1/2} \circ a'_{1/2})e_0(\overline{a_{1/2}} \circ \overline{a'_{1/2}}) \\
& \qquad = \mathrm{Re\,trace}(u^*v + v^*u)(\overline{u^*v} + \overline{v^*u}),
\end{aligned}
\tag{4.10.2}
$$

where

$$
a_{1/2} = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix} \quad \text{and} \quad a'_{1/2} = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}.
$$

Expanding (4.10.2) yields

$$
\mathrm{Re\,trace}(v\bar{u}^*\bar{v}u^* + u\bar{v}^*\bar{u}v^*) + \mathrm{Re\,trace}(v\bar{v}^*\bar{u}u^* + u\bar{u}^*\bar{v}v^*).
\tag{4.10.3}
$$

4.11. We consider the contributions of the last three summands of (4.10.1) which are perpendicular to $\mathbf{L}' \otimes \mathbb{C}$. We will denote these summands by $(\cdots)^\perp$.

In view of (4.10.3) we obtain

$$
\begin{aligned}
& \mathrm{Re\,trace}[(v\bar{v}^*)^\perp(\bar{u}u^*)^\perp + (u\bar{u}^*)^\perp(\bar{v}v^*)^\perp] \\
& \qquad - \mathrm{Re\,trace}[(u\bar{u}^* + \bar{u}u^*)^\perp \cdot (v\bar{v}^* + \bar{v}v^*)^\perp],
\end{aligned}
\tag{4.11.1}
$$

$$
\begin{aligned}
& \mathrm{Re\,trace}[(v\bar{u}^*)^\perp(\bar{v}u^*)^\perp + (u\bar{v}^*)^\perp(\bar{u}v^*)^\perp] \\
& \qquad - \mathrm{Re\,trace}[(u\bar{v}^* + \bar{v}u^*)^\perp(v\bar{u}^* + \bar{u}v^*)^\perp].
\end{aligned}
\tag{4.11.2}
$$

It is straightforward to see that here two terms cancel. Using $(A^*)^\perp = (A^\perp)^*$ we also have $\operatorname{Re}\operatorname{trace} A^\perp B^\perp = \operatorname{Re}\operatorname{trace}(A^*)^\perp(B^*)^\perp$, whence (4.11.1) and (4.11.2) finally yield

(4.11.1)′ $\quad -2\operatorname{Re}\operatorname{trace}(u\bar{u}^*)^\perp(v\bar{v}^*)^\perp,$

(4.11.2)′ $\quad -2\operatorname{Re}\operatorname{trace}(v\bar{u}^*)^\perp(v\bar{v}^*)^\perp.$

On the other hand, an explicit computation of $\mathbf{R}_6^+$ with $a_1 = x = x^*$ and $a_1' = y = y^*$ gives for the first summand

(4.11.3) $\quad \operatorname{Re}\operatorname{trace}[(x\bar{x} + u\bar{u}^*)(y\bar{y} + v\bar{v}^*) + x\bar{u}v^*\bar{y} + u^*\bar{x}y\bar{v} + u^*\bar{u}v^*\bar{v}],$

and for the second summand

(4.11.4) $\quad \operatorname{Re}\operatorname{trace}[(y\bar{x} + v\bar{u}^*)(x\bar{y} + u\bar{v}^*) + y\bar{u}u^*\bar{y} + v^*\bar{x}x\bar{v} + v^*\bar{u}u^*\bar{v}].$

We note that the first summand is also of type

(4.11.5) $\quad \operatorname{Re}\operatorname{trace}(A\bar{A}^* + u\bar{u}^*)(B\bar{B}^* + v\bar{v}^*),$

where $A = \begin{pmatrix} x & 0 \\ u^* & 0 \end{pmatrix}$ and $B = \begin{pmatrix} y & 0 \\ v^* & 0 \end{pmatrix}$.

4.12. Theorem. $\mathbf{D}_5 \geq 0$.

Proof. We first observe that all four summands of (4.11.4) are nonnegative. The last summand in (4.11.4) is

(1) $\operatorname{Re}\operatorname{trace}\bar{u}u^*\bar{v}v^*$,

and from (4.11.5) we get as one of the four nonnegative summands:

(2) $\operatorname{Re}\operatorname{trace} u\bar{u}^*v\bar{v}^*$.

Thus altogether we have

(3) $\operatorname{Re}\operatorname{trace}(u\bar{u}^*v\bar{v}^* + \bar{u}u^*\bar{v}v^*)$.

Note that here the argument satisfies $w^* = w$. On the other hand, it is easy to see that (4.11.1)′ is real, hence, subtracting (4.11.1)′ from (3) gives as argument of "Re trace" the $\mathbf{L}' \otimes \mathbb{C}$ component of the argument in (3). In view of §4.4 this gives a nonnegative summand.

Next we consider the first summand in (4.11.4). Since $x \in \operatorname{Sym}(2, \mathbb{R}) \otimes \mathbb{C}$, a straightforward computation shows $(y\bar{x})^\perp = 0$. Hence

(4) $(y\bar{x} + v\bar{u}^*)^\perp = (v\bar{u}^*)^\perp$.

As a consequence, subtracting (4.11.2)′ from the first summand of (4.11.4) results in

(5) $\operatorname{Re}\operatorname{trace}(y\bar{x} + v\bar{u}^* - 2(v\bar{u}^*)^\perp)(x\bar{y} + u\bar{v}^* - 2(u\bar{v}^*)^\perp) \geq 0$.

This finishes the proof of the theorem.

4.13. Now we consider the case of general D_6. It is easy to see that $\mathbf{U} = \mathbf{U}_1^+ + \mathbf{U}_2$ (orthogonal sum), where $\mathbf{L}$ acts as in D_6^+ on $\mathbf{U}_1$ and different in $\mathbf{U}_2$. We recall that the difference in the action of $\mathbf{L}$ on $\mathbf{U}_1$ and $\mathbf{U}_2$ consists solely of the way

$$k = \begin{pmatrix} 0 & f_3 \\ -f_3 & 0 \end{pmatrix}$$

acts, namely, like kd_1 on $\mathbf{U}_1$ and $-kd_2$ on $\mathbf{U}_2$.

As in §4.10 we see that we can assume that the number of irreducible representations of $\mathbf{L}$ in $\mathbf{U}_1$ and in $\mathbf{U}_2$ are both even, say $2n_1$ and $2n_2$. Then we consider D_6^+ constructed from $\mathbf{A}_n$, where $n = n_1 + n_2$. An inspection of Lemma 3.2 shows that $\mathbf{D}_6$ differs from $\mathbf{D}_6^+$ only in terms exclusively involving $d = d_1 + d_2$ and $d' = d_1' + d_2'$. To evaluate this we consider the map $\alpha\colon \mathbf{A}_2 \to \mathbf{A}_2$ that fixes f_0, f_1, f_2 and maps f_3 to $-f_3$. A direct calculation shows that α is, indeed, an automorphism of $\mathbf{A}_2$. Moreover, denoting by $\widetilde{F}$ the hermitian map associated with D_6 we obtain $\widetilde{F}(q_1,p_1) = F(q_1,p_1)$ and $\widetilde{F}(q_2,p_2) = \alpha F(q_2,p_2)$, $q_r,p_r \in \mathbf{U}_r$, $r = 1,2$, where F denotes the expression of D_6^+.

Note that this means $\widetilde{F}(d,d) = F(d_1,d_1) + \alpha F(d_2,d_2)$. Therefore, $\mathbf{D}_6$ and $\mathbf{D}_6^+$ only differ in contributions involving f_3 (which will be denoted by "$\cdots^\perp$" as before). Thus with $\mathbf{D}_6^+$ we have the expression

$$(4.13.1)\quad \begin{aligned} &2\{F(d_1,d_1)^\perp, F(d_1',d_1')^\perp\} + 2\{F(d_2,d_2)^\perp, F(d_2',d_2')^\perp\} \\ &\quad + 2\{F(d_1,d_1')^\perp, F(d_1',d_1)^\perp\} + 2\{F(d_2,d_2')^\perp, F(d_2',d_2)^\perp\}, \end{aligned}$$

whereas $\mathbf{D}_6$ has correspondingly

$$(4.13.2)\quad \begin{aligned} &2\{F(d_1,d_1)^\perp, F(d_1',d_1')^\perp\} - 2\{F(d_2,d_2)^\perp, F(d_2',d_2')^\perp\} \\ &\quad + 2\{F(d_1,d_1')^\perp, F(d_1',d_1)^\perp\} - 2\{F(d_2,d_2')^\perp, F(d_2',d_2)^\perp\}. \end{aligned}$$

Now we can almost verbally repeat the argument of Theorem 4.12. In the first step we no longer reduce to the $\mathbf{L}' \otimes \mathbb{C}$ component of some expression, but replace its component perpendicular to $\mathbf{L}' \otimes \mathbb{C}$ by its negative (this will not change the sign in question). The second step encounters only different factors. This shows

Theorem. $\mathbf{D}_6 \geq 0$.

4.14. In this subsection we carry out the last step to prove the main result of this paper.

Theorem. *Let $D(\Omega, F)$ be an irreducible quasisymmetric Siegel domain of rank 2. Then $D(\Omega, F)$ has nonpositive holomorphic bisectional curvature.*

Proof. Since $D(\Omega, F)$ is irreducible, $\dim \mathbf{L} \geq 3$. If $\dim \mathbf{L} = 3, 4$, then the claim follows from Theorem 4.2. If $\dim \mathbf{L} = 5$ or $\dim \mathbf{L} = 6$, then Theorem 4.12 and Theorem 4.13 prove the assertion. Assume now $\dim \mathbf{L} \geq 6$. Let $X = Y + U + Z$ and $X' = Y' + U' + Z'$ correspond to $a\partial_z + d\partial_u$ and

$a'\partial_z + d'\partial_u$ respectively. Then $\mathbf{L}'$ denotes the subspace of $\mathbf{L}$ spanned by Y, Y', Z, Z', E and $F(U, U)$. We can assume $\dim \mathbf{L}' = 6$. Since $\operatorname{rank} \mathbf{L}' = 2$, we know $\mathbf{L}' \cong \operatorname{Herm}(2; \mathbb{H})$.

Now we consider the curvature expressions of Lemma 3.2—multiplied by a negative constant. From §3.1 we know that $\{\cdot, \cdot\}$ is the complex bilinear extension of σ and $R_a = \frac{1}{2}\phi(a)$. Thus, computing the holomorphic bisectional curvature $K(X, X')$ in $D(\Omega, F)$—up to a negative multiple—we see that by the results of previous sections, the first six summands Q are just sQ' where Q' denotes the corresponding terms computed in $D(\Omega', F')$ associated with $\mathbf{L}'$. The next to the last term is

$$\begin{aligned} 2\sigma(F(U), F'(U', U')) &= 2\sigma(F'(U, U), F(U', U')) \\ &= 2\sigma(F'(U, U), F'(U', U')) \\ &= s2\sigma'(F'(U, U), F'(U', U')). \end{aligned}$$

For the last term we have

$$\begin{aligned} 2\sigma(F(U, U'), F'(U', U)) &= 2\sigma(F'(U, U'), F'(U', U)) \\ &\quad + 2\sigma(F'(U, U), F(U', U')) \\ &= s2\sigma'(F'(U, U'), F'(U', U)) + A, \end{aligned}$$

where $A = 2\sigma(F^{\perp}(U, U'), F^{\perp}(U', U)) \geq 0$. Therefore $-cK(X, X') = -c'sK'(X, X') + A$, where $A \geq 0$ and $c, c' > 0$ are some constants. Note that this does *not* contradict Lemma 4.1 because $D(\Omega', F')$ is not a submanifold of $D(\Omega, F)$. It therefore suffices to remark again that all quasisymmetric Siegel domains with $\mathbf{L} \cong \operatorname{Herm}(2, \mathbb{H})$ have nonpositive holomorphic bisectional curvatures (Theorem 4.13).

Combining the theorem above with Theorem 4.2 we thus obtain the main result of this paper.

Every quasisymmetric Siegel domain has nonpositive holomorphic bisectional curvature relative to its Bergman metric.

Added in proof. In a recent reprint, Azukawa has independently given a classification free proof of Zelow's formula for the *holomorphic* sectional curvature and has an illuminating formulation of some of the special relations of our section 2.

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